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The geometric approaches to the possible singularities in the inviscid fluid flows*

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Abstract
We consider the possible generation of singularities of a vector field transported by diffeomorphisms with derivatives of uniformly bounded determinants. We find relations between the directions of the vector field and the eigenvectors of the derivative of the back-to-label map near the singularity. We also find an invariant when we follow the motion of the integral curves of the vector field. For the 3D incompressible Euler equations these results have immediate implications about the directions of the vortex stretching and the material stretching near the possible singularities. We also have similar applications to other inviscid fluid equations such as the 2D quasi-geostrophic equation and the 3D magnetohydrodynamics equations.

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1. Vector fields transported by diffeomorphisms

1.1. Statement of the theorems

Let D be a domain in $\mathbb{R}^n$, and $T \in (0, \infty]$. Suppose that for all $t \in [0, T)$ the mapping $a \to X(a, t)$ is a diffeomorphism on D. We denote by $A(\cdot, t)$ the inverse mapping of $X(\cdot, t)$, satisfying

$$A(X(a, t), t) = a, \quad X(A(x, t), t) = x \qquad \forall a, x \in D, \quad \forall t \in [0, T).$$

In the applications to hydrodynamics in the next section the mapping $\{X(\cdot, t)\}$ is defined by a smooth velocity field $v(x, t)$ through the system of the ordinary differential equations:

$$\frac{\partial X(a, t)}{\partial t} = v(X(a, t), t); \qquad X(a, 0) = a \in D \subset \mathbb{R}^n. \tag{1.1}$$

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In such a case we say the 'particle trajectory' map $X(\cdot, t)$ and its inverse, the 'back-to-label' map $A(\cdot, t)$ are generated by the fluid velocity field $v(x, t)$.

Definition 1.1. *We say that a parameterized vector field* $W(\cdot, \cdot) : D \times [0, T) \to \mathbb{R}^n$ *is transported by a differentiable mapping* $X(\cdot, t)$ *from* D *into itself for all* $t \in [0, T)$ *if*

$$W(X(a, t), t) = \nabla_a X(a, t) W_0(a) \tag{1.2}$$

holds for all $(a, t) \in D \times [0, T)$, *where we set* $W_0(x) = W(x, 0)$.

We note that (1.2) corresponds to the well-known vorticity transport formula for the incompressible Euler equations. Actually it is well known (see e.g. [32]) that (1.2) is equivalent to saying that the vector field $W(x, t)$ satisfies the system of differential equations:

$$\begin{cases} \dfrac{\partial W}{\partial t} + (v \cdot \nabla) W = (W \cdot \nabla) v, \\ W(x, 0) = W_0(x) \end{cases} \tag{1.3}$$

on $D \times [0, T)$, where $v(x, t)$ is defined from $X(\cdot, t)$ by (1.1). In this paper we are concerned with the study of the direction of $W(x, t)$ and the directions of stretching/compressions induced by a 'generalized volume preserving' mapping $X(\cdot, t)$ near possible singularities in (1.2) (or, equivalently in (1.3)). This will be done efficiently in terms of the derivative of its inverse mapping $A(\cdot, t) = X^{-1}(\cdot, t)$. The main motivation of the current study is to understand the dynamic relation between the vortex stretching and the material stretching when we approach possible singularities in the 3D incompressible Euler equations and other inviscid flows.

Theorem 1.1. *Let* $W(x, t)$ *be a vector field on* $D \subset \mathbb{R}^n$ *defined for* $t \in [0, T)$. *We set* $W_0(x) = W(x, 0)$ *with* $\|W_0\|_{L^\infty(D)} < \infty$. *Suppose* $W(x, t)$ *is transported by a diffeomorphism* $\{X(\cdot, t)\}_{t \in [0,T)}$ *on* $D \subset \mathbb{R}^n$, *whose inverse is* $A(\cdot, t)$. *We assume that*

$$\sup_{(a,t) \in D \times [0,T)} |\det(\nabla_a X(a, t))| < \infty. \tag{1.4}$$

Let us set by $\{(\lambda_j, e_j)\}_{j=1}^n$ *the eigenvalue and eigenvector pairs of the symmetric, positive definite matrix*

$$M(x, t) = (\nabla A(x, t))^* \nabla A(x, t)$$

with the order of magnitude

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0. \tag{1.5}$$

Suppose there exists a sequence (x_k, t_k) *and* $(\bar{x}, \bar{t})$ *in* $\bar{D} \times [0, T]$ *such that* $\lim_{k \to \infty}(x_k, t_k) = (\bar{x}, \bar{t})$ *and*

$$\lim_{k \to \infty} |W(x_k, t_k)| = \infty.$$

Then, necessarily

$$\lim_{k \to \infty} \lambda_1(x_k, t_k) = \infty, \qquad \text{and} \qquad \lim_{k \to \infty} \lambda_n(x_k, t_k) = 0. \tag{1.6}$$

Let m *be the largest number in* $\{1, \ldots, n\}$ *such that*

$$\lim_{k \to \infty} \lambda_j(x_k, t_k) > 0 \quad \forall j \in \{1, \ldots, m\}. \tag{1.7}$$

We set by $\Xi(x, t)$ *the direction field of* $W(x, t)$ *defined by*

$$\Xi(x, t) = \frac{W(x, t)}{|W(x, t)|} \qquad \text{whenever} \quad |W(x, t)| \neq 0.$$

Then,

$$\lim_{k\to\infty} e_j(x_k,t_k)\cdot\Xi(x_k,t_k)=0\quad \forall j\in\{1,\ldots,m\}. \tag{1.8}$$

Remark 1.1. Let $v\in\mathbb{R}^n$. Then, the quantity $|\nabla_a X(a,t)v|/|v|$ has the meaning of the rate of stretching(compression) in the direction of v induced by the trajectory mapping $X(\cdot,t)$ if the quantity is more(less) than 1. Indeed, let $\{\gamma_0(s)\}_{s\in(-\varepsilon,\varepsilon)}$ be a curve in $\mathbb{R}^n$ such that

$$\gamma_0(0)=a,\qquad \left.\frac{\partial\gamma_0(s)}{\partial s}\right|_{s=0}=v.$$

We set $X(\gamma_0(s),t)=\gamma(s,t)$. Then,

$$\frac{\partial\gamma(s,t)}{\partial s}=\nabla_a X(\gamma_0(s),t)\frac{\partial\gamma_0(s)}{\partial s}, \tag{1.9}$$

and

$$\frac{|\nabla_a X(a,t)v|}{|v|}=\left.\frac{\left|\nabla_a X(\gamma_0(s),t)\frac{\partial\gamma_0(s)}{\partial s}\right|}{\left|\frac{\partial\gamma_0(s)}{\partial s}\right|}\right|_{s=0}=\left.\frac{\left|\frac{\partial\gamma(s,t)}{\partial s}\right|}{\left|\frac{\partial\gamma_0(s)}{\partial s}\right|}\right|_{s=0}=\left.\left|\frac{\partial\gamma(s,t)}{\partial\gamma_0(s)}\right|\right|_{s=0}, \tag{1.10}$$

which provides us with the desired interpretation. By the Rayleigh–Ritz theorem [25] and the fact that $X(\cdot,t)$ is a diffeomorphism we have

$$\begin{aligned}\lambda_1(x,t)&=\max\{\lambda_1(x,t),\ldots,\lambda_n(x,t)\}\\ &=\sup_{v\neq0}\frac{v^*M(x,t)v}{|v|^2}=\sup_{v\neq0}\frac{|\nabla A(x,t)v|^2}{|v|^2}\\ &=\sup_{w\neq0}\frac{|w|^2}{|\nabla_a X(a,t)w|^2}=\frac{1}{\inf_{w\neq0}\frac{|\nabla_a X(a,t)w|^2}{|w|^2}}.\end{aligned}$$

Hence,

$$\inf_{v\neq0}\frac{|\nabla_a X(a,t)v|}{|v|}=\frac{1}{\sqrt{\lambda_1(x,t)}}, \tag{1.11}$$

where $x=X(a,t)$. Similarly, for $\lambda_n(x,t)=\min\{\lambda_1(x,t),\ldots,\lambda_n(x,t)\}$, we obtain

$$\sup_{v\neq0}\frac{|\nabla_a X(a,t)v|}{|v|}=\frac{1}{\sqrt{\lambda_n(x,t)}} \tag{1.12}$$

with $x=X(a,t)$. In particular, in the case of $\det(\nabla_a X(a,t))\equiv1$ (incompressible flow), the quantity $1/\sqrt{\lambda_1(x,t)}(<1)$ has the meaning of the minimum compression rate, while $1/\sqrt{\lambda_n(x,t)}(>1)$ has the meaning of the maximum stretching rate at (x,t), except the case $\nabla_a X(a,t)=I$, where $a=A(x,t)$.

Remark 1.2. In particular (1.6) implies that the directions of the infinite stretching rate and the zero compression rate are mutually orthogonal to each other.

Remark 1.3. Since $\lim_{k\to\infty}\lambda_j(x_k,t_k)=0$ for all $j=m+1,\ldots,n$ by the hypothesis of the above theorem, the conclusion (1.8) implies that as $(x_k,t_k)\to(\bar{x},\bar{t})$ the sequence of direction vectors $\{\Xi(x_k,t_k)\}$ tends to be on the linear span generated by the vectors with the directions of infinite stretching rates.

The first part of the following theorem could be regarded as a generalization of the well-known Helmholtz vortex theorem for the incompressible Euler equations [24].

Theorem 1.2. *Suppose $W(x,t)$ is a vector field transported by a diffeomorphism $\{X(\cdot,t)\}$, $t\in[0,T)$. Let $\{\gamma_0(s)\}_{s\in I}$ be an integral curve of $W(x,0)$, then $\gamma(s,t):=X(\gamma_0(s),t)$ is also an integral curve of $W(\gamma(s,t),t)$. Moreover, we have the following invariant:*

$$\frac{|W(\gamma(s,t),t)|}{\left|\frac{\partial\gamma(s,t)}{\partial s}\right|}=\frac{|W_0(\gamma_0(s))|}{\left|\frac{\partial\gamma_0(s)}{\partial s}\right|}. \tag{1.13}$$

Remark 1.4. We will see in the proof of the above theorem that the invariant (1.13) is due to the fact that the integral curve of a vector field has re-parametrization symmetry.

1.2. Proof of the theorems

Proof of Theorem 1.1. The vector field transport formula

$$W(X(a,t),t)=\nabla_a X(a,t)W_0(a)$$

can be written as

$$\nabla A(x,t)W(x,t)=W_0(A(x,t)) \tag{1.14}$$

in terms of $A(x,t)=X^{-1}(x,t)$. Hence,

$$\begin{aligned}|W_0(A(x,t))|^2&=W(x,t)^*(\nabla A(x,t))^*\nabla A(x,t)W(x,t)\\&=|W(x,t)|^2\,\Xi(x,t)^*M(x,t)\Xi(x,t)\\&=|W(x,t)|^2\left(\lambda_1(x,t)\widetilde{\Xi}_1^2(x,t)+\cdots+\lambda_n(x,t)\widetilde{\Xi}_n^2(x,t)\right),\end{aligned} \tag{1.15}$$

where we set

$$\widetilde{\Xi}(x,t)=O(x,t)\Xi(x,t),\qquad O^*MO=\mathrm{diag}(\lambda_1,\ldots,\lambda_n).$$

Namely, the $n\times n$ orthogonal matrix $O(x,t)$ diagonalizes the positive definite, symmetric matrix M. By the hypothesis (1.4),

$$\begin{aligned}\Lambda&:=\inf_{(x,t)\in D\times[0,T)}[\lambda_1(x,t)\ldots\lambda_n(x,t)]\\&=\inf_{(x,t)\in D\times[0,T)}\det M=\inf_{(x,t)\in D\times[0,T)}\det(\nabla A(x,t))^2\\&=\frac{1}{\sup_{(x,t)\in D\times[0,T)}\det(\nabla_a X(a,t))\big|^2_{a=A(x,t)}}>0.\end{aligned} \tag{1.16}$$

By definition

$$\widetilde{\Xi}_1^2(x,t)+\cdots+\widetilde{\Xi}_n^2(x,t)=|O(x,t)\Xi(x,t)|^2=|\Xi(x,t)|^2=1. \tag{1.17}$$

Hence, from (1.15) and the inequality $\frac{a_1+\cdots+a_n}{n}\geqslant(a_1\ldots a_n)^{\frac1n}$ for $a_1,\ldots,a_n\geqslant0$, we obtain that

$$\begin{aligned}\frac{\|W_0\|^2_{L^\infty}}{|W(x,t)|^2}&\geqslant\lambda_1(x,t)\widetilde{\Xi}_1^2(x,t)+\cdots+\lambda_n(x,t)\widetilde{\Xi}_n^2(x,t)\\&\geqslant n\left(\lambda_1(x,t)\widetilde{\Xi}_1^2(x,t)\ldots\lambda_n(x,t)\widetilde{\Xi}_n^2(x,t)\right)^{\frac1n}\\&=n\left(\lambda_1(x,t)\ldots\lambda_n(x,t)\right)^{\frac1n}\left(\widetilde{\Xi}_1^2(x,t)\ldots\widetilde{\Xi}_n^2(x,t)\right)^{\frac1n}\\&\geqslant n\Lambda^{\frac1n}\left|\widetilde{\Xi}_1(x,t)\ldots\widetilde{\Xi}_n(x,t)\right|^{\frac2n}.\end{aligned} \tag{1.18}$$

Let $\{(x_k,t_k)\}$ be a sequence such that $(x_k,t_k)\to(\bar{x},\bar{t})$ as $k\to\infty$, and

$$\lim_{k\to\infty}|W(x_k,t_k)|=\infty.$$

Then, the first inequality of (1.18) implies that

$$\lim_{k\to\infty} \lambda_j(x_k, t_k)\widetilde{\Xi}_j^2(x_k, t_k) = 0 \qquad \forall j = 1, \dots, n. \tag{1.19}$$

From (1.17) we find that there exists $j_0 \in \{1, \dots, n\}$ such that

$$\lim\inf_{k\to\infty} |\widetilde{\Xi}_{j_0}(x_k, t_k)| \geqslant \frac{1}{\sqrt{n}}. \tag{1.20}$$

Hence, from (1.5) we have

$$\lim_{k\to\infty} \lambda_n(x_k, t_k) \leqslant \lim_{k\to\infty} \lambda_{j_0}(x_k, t_k) = 0, \tag{1.21}$$

which, in turn, implies by (1.16) and (1.8) that

$$\lim_{k\to\infty} \lambda_1(x_k, t_k) = \infty. \tag{1.22}$$

Thus we find there exists $m \in \{1, \dots, n-1\}$ satisfying (1.7). Now (1.19) and (1.7) imply that

$$\lim_{k\to\infty} \widetilde{\Xi}_j(x_k, t_k) = \lim_{k\to\infty} [O(x_k, t_k)\Xi(x_k, t_k)]_j = \lim_{k\to\infty} e_j(x_k, t_k)\cdot\Xi(x_k, t_k) = 0$$

for all $j \in \{1, \dots, m\}$. □

Proof of Theorem 1.2. Taking derivative of $\gamma(s,t) = X(\gamma_0(s), t)$ with respect to $s \in I$, we have

$$\frac{\partial\gamma(s,t)}{\partial s} = \nabla_a X(\gamma_0(s), t)\frac{\partial\gamma_0(s)}{\partial s}. \tag{1.23}$$

Since $W(x,t)$ is transported by $\{X(\cdot,t)\}$, we have, along the curve $t \mapsto \gamma(s,t)$,

$$W(\gamma(s,t), t) = \nabla_a X(\gamma_0(s), t)W_0(\gamma_0(s)). \tag{1.24}$$

By hypothesis, since $\gamma_0(s)$ is an integral curve of $W_0(\gamma_0(s))$, there exists $f(s) \neq 0$ for all $s \in I$ such that

$$\frac{\partial\gamma_0(s)}{\partial s} = f(s)W_0(\gamma_0(s)), \tag{1.25}$$

and from (1.23) we have

$$\frac{\partial\gamma(s,t)}{\partial s} = f(s)\nabla_a X(\gamma_0(s), t)W_0(\gamma_0(s)) = f(s)W(\gamma(s,t), t), \tag{1.26}$$

which shows that $s \mapsto \gamma(s,t)$ is an integral curve of $W(\gamma(s,t), t)$ for each $t \in [0, T)$. From (1.25) and (1.26) we obtain

$$\frac{1}{|f(s)|} = \frac{|W(\gamma(s,t), t)|}{\left|\frac{\partial\gamma}{\partial\sigma}(s,t)\right|} = \frac{|W_0(\gamma_0(s)|}{\left|\frac{\partial\gamma_0}{\partial\sigma}(s)\right|}. \tag{1.27}$$

□

2. Applications to inviscid hydrodynamics

We discuss the implications of the previous general theorems on some of the ideal fluid mechanics equations.

2.1. The surface quasi-geostrophic equation

In this subsection we are concerned with the the following 2D quasi-geostrophic equation in $\mathbb{R}^2$:

$$(\mathrm{QG}) \begin{cases} \dfrac{\partial\theta}{\partial t} + (v\cdot\nabla)\theta = 0, \\ v = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta, \\ \theta(x,0) = \theta_0(x), \end{cases}$$

where $\theta = \theta(x_1, x_2, t)$ denotes the scalar temperature, $v = (v_1, v_2)$, $v_j = v_j(x_1, x_2, t)$, $j = 1, 2$, is the velocity of the fluid, and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. Thanks to the pioneering work by Constantin, Majda and Tabak [14], in particular the observation of its resemblance to the 3D Euler equations, there are many studies on (QG) (see, e.g. [10, 15, 16, 18, 33, 34] and references therein). Let $\{X(\cdot, t)\}$ be the particle trajectory mapping generated by $v(x, t)$. Taking operation of $\nabla^\perp$ on the first equation of (QG) we obtain

$$\frac{\partial}{\partial t}\nabla^\perp\theta + (v\cdot\nabla)\nabla^\perp\theta = (\nabla^\perp\theta\cdot\nabla)v, \tag{2.1}$$

from which we have the transport formula for $\nabla^\perp\theta(x, t)$,

$$\nabla^\perp\theta(X(a,t),t) = \nabla_a X(a,t)\nabla^\perp\theta_0(a). \tag{2.2}$$

As in the previous section we set the back-to-label map, $A(\cdot, t) = X^{-1}(\cdot, t)$ below. The following theorem is immediate from (2.2) and theorem 1.1, and the fact that det $(\nabla_a X(a, t)) \equiv 1$, which is equivalent to the incompressibility condition, div $v = 0$.

Theorem 2.1. *Let $(v(x, t), \theta(x, t))$ be a smooth solution of (QG) with initial data satisfying $\|\nabla\theta_0\|_{L^\infty} < \infty$, which generates the particle trajectory map $\{X(\cdot, t)\}$ and the back-to-label map $A(\cdot, t)$. We set the direction vector field $\xi(x, t) = \frac{\nabla^\perp\theta(x,t)}{|\nabla^\perp\theta(x,t)|}$, and let $\{e_1(x, t), e_2(x, t)\}$ and $\{\lambda_1(x, t), \lambda_2(x, t)\}$ be the normalized eigenvectors and the corresponding eigenvalues of the matrix*

$$M(x,t) = (\nabla A(x,t))^T\,\nabla A(x,t).$$

We keep the order of magnitude such that

$$\lambda_1(x,t) > \lambda_2(x,t) > 0 \qquad \forall (x,t).$$

Suppose there exists a sequence $\{(x_k, t_k)\}$ tending to $(\bar{x}, \bar{t})$ as $k \to \infty$ such that $\lim_{k\to\infty} |\nabla\theta(x_k, t_k)| = \infty$, then necessarily

$$\lim_{k\to\infty}\lambda_1(x_k,t_k) = \infty, \qquad \lim_{k\to\infty}\lambda_2(x_k,t_k) = 0, \tag{2.3}$$

and

$$\lim_{k\to\infty}|\xi(x_k,t_k) - e_2(x_k,t_k)| = 0. \tag{2.4}$$

We just note that (2.4) follows from

$$\lim_{k\to\infty}\xi(x_k,t_k)\cdot e_1(x_k,t_k) = 0 \tag{2.5}$$

together with $(\xi\cdot e_1)^2 + (\xi\cdot e_2)^2 = |\xi|^2 = 1$. On the other hand hand, (2.4) implies that the direction field tends to align with the direction of the infinite stretching rate near the possible singularity, while (2.5) shows that the direction of zero compression rate is orthogonal to it.

Since any smooth level curve of θ_0 is an integral curve of $\nabla^\perp\theta_0$, applying theorem 1.2 to (QG), we obtain the following theorem.

Theorem 2.2. *Let $(\theta(x,t).v(x,t))$ be a smooth solution of (QG), and $\{X(\cdot,t)\}$ the particle trajectory generated by $v(x,t)$. Let $\{\gamma_0(s)\}_{s\in I}$ be a level curve of θ_0. We set $\gamma(s,t) = X(\gamma_0(s),t)$, then $\gamma(s,t)$ is also a level curve of $\theta(x,t)$. Moreover, we have the following invariants along the trajectories of level curves of $\theta(x,t)$:*

$$\frac{|\nabla^\perp\theta(\gamma(s,t),t)|}{\left|\frac{\partial\gamma(s,t)}{\partial s}\right|} = \frac{|\nabla^\perp\theta_0(\gamma_0(s))|}{\left|\frac{\partial\gamma_0(s)}{\partial s}\right|}. \tag{2.6}$$

Corollary 2.1. *Suppose there exist a sequence $\{(s_k,t_k)\}$ and $(\bar{s},\bar{t})$ such that $(s_k,t_k)\to(\bar{s},\bar{t})$, and*

$$\lim_{k\to\infty} |\nabla^\perp\theta(\gamma(s_k,t_k),t_k)| = \infty, \tag{2.7}$$

then necessarily

$$\lim_{k\to\infty} \left|\frac{\partial\gamma}{\partial s}(s_k,t_k)\right| = \infty. \tag{2.8}$$

Namely, the blow-up of $|\nabla^\perp\theta|$ at a point is accompanied by an infinite stretching of level curves at the same point in the tangential direction to the curve.

2.2. The Euler equations for isentropic flows

We are concerned here with the following Euler equations for the isentropic fluid flows in $\mathbb{R}^n$, $n = 2, 3$,

$$(E)\begin{cases} \rho\dfrac{\partial v}{\partial t} + \rho(v\cdot\nabla)v = -\nabla p, \\ \dfrac{\partial\rho}{\partial t} + \operatorname{div}(\rho v) = 0, \\ v(x,0) = v_0(x), \qquad \rho(x,0) = \rho_0(x), \end{cases}$$

where $v = (v_1,\dots,v_n)$, $v_j = v_j(x,t)$, $j = 1,\dots,n$, is the velocity of the flow, $\rho = \rho(x,t)$ is the mass density of the fluid, $p = p(x,t)$ is the scalar pressure, and v_0, ρ_0 are the given initial velocity and density. The homogeneous incompressible Euler equations corresponds to $\rho(x,t) \equiv$ const., for which we denote by $(E)_0$. The problems of finite time blow-up/global regularity for the systems (E) and $(E)_0$ are both outstanding open problems in the mathematical fluid mechanics. For $(E)_0$ there are results on the blow-up criterion initiated by Beale, Kato and Majda [2], and refined by authors in [4, 10, 12, 28, 29, 31]). The study of the Euler system in terms of the volume preserving maps was previously done by many authors(see e.g. [1, 3]). The geometric-type approaches emphasizing the role of the direction of vorticity for the regularity/singularity of solutions are studied in [13, 19, 20, 23, 6], the spectral dynamics-type approaches are studied in [30, 5], and some of the plausible scenarios leading to singularities are excluded in [7, 8, 17, 18]. Let $\{X(\cdot,t)\}$ be the particle trajectory mapping generated by $v(x,t)$, defined by a smooth solution of the solutions of (E), and $A(x,t) = X^{-1}(x,t)$ be the back-to-label map. Let $\omega(x,t) = \operatorname{curl} v(x,t)$ be the vorticity. The following vorticity transport formula is well known (see e.g. [9]) for (E):

$$\frac{\omega(X(a,t),t)}{\rho(X(a,t),t)} = \nabla_a X(a,t)\frac{\omega_0(a)}{\rho_0(a)}. \tag{2.9}$$

Applying theorem 1.1 to the case of $(E)_0$, for which we have

$$\omega(X(a,t),t) = \nabla_a X(a,t)\omega_0(a), \tag{2.10}$$

as well as $\det(\nabla_a X(a,t)) \equiv 1$, we obtain the following theorem.

Theorem 2.3. *Let $\omega(x,t)$ be the vorticity of a smooth solution $v(x,t)$ of $(E)_0$ in $\mathbb{R}^3$ with initial vorticity satisfying $\|\omega_0\|_{L^\infty} < \infty$. The particle trajectory map $\{X(\cdot,t)\}$ and the particle trajectory map $A(\cdot,t)$ are generated by $v(x,t)$. We set the vorticity direction field $\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}$. Let $\{(\lambda_j(x,t), e_j(x,t))\}_{j=1}^3$ be the pairs of the eigenvalues and normalized eigenvectors of the matrix*

$$M(x,t) = (\nabla A(x,t))^T \, \nabla A(x,t),$$

where we keep the ordering for the corresponding eigenvalues

$$\lambda_1(x,t) \geqslant \lambda_2(x,t) \geqslant \lambda_3(x,t) > 0. \tag{2.11}$$

Suppose there exists a sequence $\{(x_k,t_k)\}$ tending to $(\bar{x},\bar{t})$ as $k \to \infty$ such that $\lim_{k\to\infty} |\omega(x_k,t_k)| = \infty$, then necessarily

$$\lim_{k\to\infty} \lambda_1(x_k,t_k) = \infty \qquad \text{and} \qquad \lim_{k\to\infty} \lambda_3(x_k,t_k) = 0, \tag{2.12}$$

and

$$\lim_{k\to\infty} \xi(x_k,t_k) \cdot e_1(x_k,t_k) = 0. \tag{2.13}$$

Furthermore, if

$$\lim\inf_{k\to\infty} \lambda_2(x_k,t_k) > 0, \tag{2.14}$$

then

$$\lim_{k\to\infty} \xi(x_k,t_k) \cdot e_2(x_k,t_k) = 0. \tag{2.15}$$

Remark 2.1. In the case when (2.13) and (2.15) happen, we note that

$$\lim_{k\to\infty} \xi(x_k,t_k) \cdot e_3(x_k,t_k) = 1, \tag{2.16}$$

which is equivalent to

$$\lim_{k\to\infty} |\xi(x_k,t_k) - e_3(x_k,t_k)| = 0. \tag{2.17}$$

Namely, as (x_k,t_k) tends to $(\bar{x},\bar{t})$, the sequence of vorticity direction vectors $\{\xi(x_k,t_k)\}$ tends to align with the eigenvector of $M(x_k,t_k)$ with the smallest eigenvalue, which is in the direction of maximum stretching rate. Taking into account formula (2.9), we obtain the following theorem immediately from theorem 1.2.

Theorem 2.4. *Let $(v(x,t), \rho(x,t))$ be a smooth solution of (E) and $\{X(\cdot,t)\}$ be the particle trajectory generated by $v(x,t)$. Let $\gamma_0(s)$ be a vortex line for the initial vorticity ω_0. We set $\gamma(s,t) = X(\gamma_0(s),t)$, which is also a vortex line by the Helmholtz theorem. Then, we have the following invariants along the trajectories of the vortex lines:*

$$\frac{|\omega(\gamma(s,t),t)|}{|\rho(\gamma(s,t),t)|\left|\frac{\partial\gamma(s,t)}{\partial s}\right|} = \frac{|\omega_0(\gamma_0(s))|}{|\rho_0(\gamma_0(s))|\left|\frac{\partial\gamma_0(s)}{\partial s}\right|}. \tag{2.18}$$

Corollary 2.2. *Let $\omega = \operatorname{curl} v$ and $\gamma(s,t)$ as in theorem 2.4 and $\|\omega_0/\rho_0\|_{L^\infty} < \infty$. Suppose there exist a sequence $\{(s_k,t_k)\}$ and $(\bar{s},\bar{t})$ such that $(s_k,t_k) \to (\bar{s},\bar{t})$, and*

$$\lim_{k\to\infty} \frac{|\omega(\gamma(s_k,t_k),t_k)|}{|\rho(\gamma(s_k,t_k),t_k)|} = \infty, \tag{2.19}$$

then necessarily

$$\lim_{k\to\infty} \left|\frac{\partial\gamma}{\partial s}(s_k,t_k)\right| = \infty. \tag{2.20}$$

Namely, a singularity of $|\omega|/|\rho|$ at a point is accompanied by infinite stretching of the vortex line at the same point.

Remark 2.2. In the case of $(E)_0$ thin vortex tube (vortex filament) stretching near the singularity of vorticity in the 3D Euler equations is a well-known fact in the elementary fluid mechanics (see e.g.[9]), which is immediate from Kelvin's circulation theorem. The above corollary, in contrast, is about stretching of individual vortex lines, not the tubes.

2.3. The magnetohydrodynamic equations

We are concerned here with the ideal MHD system in $\mathbb{R}^3$,

$$(\text{MHD}) \begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p - b\times \operatorname{curl} b, \\ \dfrac{\partial b}{\partial t} + (v\cdot\nabla)b = (b\cdot\nabla)v, \\ \quad \operatorname{div} v = \operatorname{div} b = 0, \\ v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \end{cases}$$

where $v = (v_1, v_2, v_3)$, $v_j = v_j(x,t)$, $j = 1, 2, 3$, is the velocity of the flow, $p = p(x,t)$ is the scalar pressure, $b = (b_1, b_2, b_3)$, $b_j = b_j(x,t)$, is the magnetic field, and v_0, b_0 are the given initial velocity and magnetic field, satisfying div $v_0 = \operatorname{div} b_0 = 0$. Below, $\{X(a,t)\}$ is the particle trajectory mapping generated by $v(x,t)$, defined by a smooth solution of (MHD), and $A(x,t) = X^{-1}(x,t)$ is the back-to-label map. As for the vorticity transport formula the second equation of (MHD) implies that we have

$$b(X(a,t),t) = \nabla_a X(a,t) b_0(a), \tag{2.21}$$

which provides us with the following theorem due to theorem 1.2.

Theorem 2.5. *Let $v(x,t)$ be a smooth solution of (MHD) with the initial data satisfying $\|b_0\|_{L^\infty} < \infty$, which generates the particle trajectory map $\{X(\cdot,t)\}$ and the particle trajectory map $A(\cdot,t)$. We set the direction-vector field of the magnetic field $\xi(x,t) = \frac{b(x,t)}{|b(x,t)|}$, and let $\{(\lambda_j(x,t), e_j(x,t))\}_{j=1}^3$ be the pairs of the eigenvalues and normalized eigenvectors of the matrix*

$$M(x,t) = (\nabla A(x,t))^T \nabla A(x,t),$$

where we keep the ordering for the corresponding eigenvalues

$$\lambda_1(x,t) \geqslant \lambda_2(x,t) \geqslant \lambda_3(x,t) > 0. \tag{2.22}$$

Suppose there exists a sequence $\{(x_k, t_k)\}$ tending to $(\bar{x}, \bar{t})$ as $k \to \infty$ such that $\lim_{k\to\infty} |b(x_k, t_k)| = \infty$, then necessarily

$$\lim_{k\to\infty} \lambda_1(x_k, t_k) = \infty, \qquad \textit{and} \qquad \lim_{k\to\infty} \lambda_3(x_k, t_k) = 0, \tag{2.23}$$

and

$$\lim_{k\to\infty} \xi(x_k, t_k)\cdot e_1(x_k, t_k) = 0. \tag{2.24}$$

Furthermore, if

$$\liminf_{k\to\infty} \lambda_2(x_k, t_k) > 0, \tag{2.25}$$

then

$$\lim_{k\to\infty} \xi(x_k, t_k)\cdot e_2(x_k, t_k) = 0. \tag{2.26}$$

Remark 2.3. We have a similar remark to remark 2.1, and have

$$\lim_{k\to\infty} |\xi(x_k, t_k) - e_3(x_k, t_k)| = 0, \tag{2.27}$$

in case (2.25) holds. Namely, as (x_k, t_k) tends to $(\bar{x}, \bar{t})$, the sequence of magnetic direction vectors $\{\xi(x_k, t_k)\}$ tends to align with the eigenvector of $M(x_k, t_k)$ with the smallest eigenvalue, which is in the direction of maximum stretching rate.

Theorem 2.6. *Let $(v(x,t), b(x,t))$ be a smooth solution of (MHD), and $\{X(\cdot, t)\}$ the particle trajectory generated by $v(x,t)$. Let $\gamma_0(s)$ be an integral curve for the initial magnetic field $b_0(\gamma_0(s))$. We set $\gamma(s,t) = X(\gamma_0(s), t)$, which is also an integral curve of the magnetic field $b(x,t)$. Then, we have the following invariants:*

$$\frac{|b(\gamma(s,t),t)|}{\left|\frac{\partial \gamma(s,t)}{\partial s}\right|} = \frac{|b_0(\gamma_0(s))|}{\left|\frac{\partial \gamma_0(s)}{\partial s}\right|}. \tag{2.28}$$

Corollary 2.3. *Let $b, \gamma(s,t)$ be as in theorem 2.6 and $\|b_0\|_{L^\infty} < \infty$. Suppose there exist a sequence $\{(s_k, t_k)\}$ and $(\bar{s}, \bar{t})$ such that $(s_k, t_k) \to (\bar{s}, \bar{t})$, and*

$$\lim_{k\to\infty} |b(\gamma(s_k, t_k), t_k)| = \infty, \tag{2.29}$$

then necessarily

$$\lim_{k\to\infty} \left|\frac{\partial \gamma}{\partial s}(s_k, t_k)\right| = \infty. \tag{2.30}$$

Namely, a singularity of the magnetic field of (MHD) is accompanied by an infinite stretching of magnetic field lines in the direction of magnetic field.

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